Scale Vs. Conformal Invariance in the AdS/CFT Correspondence

Adel M. Awad♯ and Clifford V. Johnson♮

♯*Department of Physics and Astronomy, University of Kentucky, Lexington, KY 40506, U.S.A.*
and
Department of Physics, Faculty of Science, Ain Shams University, Cairo 11566, Egypt

♮*Centre for Particle Theory, Department of Mathematical Sciences, University of Durham, Durham, DH1 3LE, U.K.*

adel@pa.uky.edu, c.v.johnson@durham.ac.uk

(June 2000)

We present two examples of non–trivial field theories which are scale invariant, but not conformally invariant. This is done by placing certain field theories, which are conformally invariant in flat space, onto curved backgrounds of a specific type. We define this using the AdS/CFT correspondence, which relates the physics of gravity in asymptotically Anti–de Sitter (AdS) spacetimes to that of a conformal field theory (CFT) in one dimension fewer. The AdS rotating (Kerr) black holes in five and seven dimensions provide us with the examples, since by the correspondence we are able to define and compute the action and stress tensor of four and six dimensional field theories residing on rotating Einstein universes, using the "boundary counterterm" method. The rotation breaks conformal but not scale invariance. The AdS/CFT framework is therefore a natural arena for generating such examples of non–trivial scale invariant theories which are not conformally invariant.

There is an often quoted piece of folklore in the subject that states that any non–trivial example of a field theory which is scale invariant is automatically conformally invariant. Proofs of this statement only exist in certain specific situations, and in fact it is known to be not generally applicable. Some discussion can be found in *e.g.,* refs. [1–5]. The focus in those cases is on finding field theory examples in flat space. Another way to violate conformal invariance is of course to place the field theory on a spacetime with non–zero curvature. Then there are conformal anomalies, and the non–vanishing trace of the stress–energy tensor, $\widehat{T}_{ab}$, can be written in terms of various local curvature invariants of the background spacetime. The general form of the anomaly in dimension n (which is even, since we only have conformal anomalies in those cases) is given by (see *e.g.* ref. [6]):

$$\widehat{T}^a_a = \mathrm{c}_0 E_n + \sum_i \mathrm{c}_i I_i + \nabla_a J^a \ . \tag{1}$$

Here, the c's are constants, E_n is the Euler density, I_i are terms constructed from the Weyl tensor and its derivatives, and the last term is a collection of total derivative terms. The first type of term is called "type A", the next "type B", and the last "type D". It is important to note that the coefficients of all terms are regularisation scheme independent *except* the type D anomaly. These latter terms can be removed by a suitable addition of local counterterms to the action. The type A anomaly is only locally a total derivative, in general.

In order to construct a non–trivial example of a scale invariant theory which is not conformally invariant, we can simply place a conformally invariant theory on a spacetime $\mathcal{S}$ for which the type A anomaly does not vanish, while the types B and D anomalies do. In this case, there will be an irremovable anomaly $\widehat{T}^a_a$, for which $\int_{\mathcal{S}} \widehat{T}^a_a = 0$, showing that scale invariance is preserved.

In short, we must find a way to place a conformally invariant theory on a spacetime for which the Euler density is non–vanishing, but which is topologically trivial, so that the integral vanishes. Our spacetime must also be conformally flat, thus not contributing to the Type B anomaly, which would break scale invariance too. In this letter, we show how to do this, and in this way find a new class of counterexamples to the folklore.

New tools have appeared on the market for defining and studying conformally invariant field theories in interesting situations, often even at strong coupling. One of these, the "AdS/CFT correspondence", relates an $(n+1)$–dimensional theory of gravity on anti–de Sitter (AdS) spacetime (times a compact manifold) to a conformal field theory (CFT) in n dimensions. This duality arose as a result of investigating [7] a large number, N, of parallel D3–branes (reviewed in refs. [8]) in the context of the low energy, classical limit of type IIB superstring theory, —the supergravity limit— on five dimensional anti–de Sitter spacetime times a five sphere ($\mathrm{AdS}_5\times S^5$). The dual CFT in this case is the four dimensional $\mathcal{N}=4$ supersymmetric $SU(N)$ Yang–Mills theory for large N. For other dimensions, the dual theories exist, but are less well understood. For example eleven dimensional supergravity on $\mathrm{AdS}_7\times S^4$ is dual to a six dimensional "(0,2)" CFT, (the notation denoting the number and chirality of the six dimensional supercharges), also at large N. (See ref. [9] for a review.) A precise statement of the AdS/CFT correspondence [10,11] equates the partition

functions:

$$Z_{AdS}(\phi_i) = Z_{CFT}(\phi_{0,i}) \ . \tag{2}$$

From the gravity–on–AdS point of view, ϕ_i is a bulk field constrained to the values $\phi_{0,i}$ on the AdS boundary, while from the CFT point of view, $\phi_{0,i}$ are sources for pointlike operators, $\mathcal{O}_i$, in the theory. In the low energy limit of the theory one can use the classical gravitational action to calculate the partition function of the CFT "on the boundary". This action has the form [12],

$$\begin{aligned} I_{\rm bulk} + I_{\rm surf} = & -\frac{1}{16\pi G}\int_{\mathcal{M}} d^{n+1}x\sqrt{-g}\left(R + \frac{n(n-1)}{l^2}\right) \\ & -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^n x\sqrt{-h}\,K. \end{aligned} \tag{3}$$

The first term is the Einstein–Hilbert action with negative cosmological constant ($\Lambda{=}-n(n-1)/2l^2$). The second term is the Gibbons–Hawking boundary term. Here, h_{ab} is the boundary metric and K is the trace of the extrinsic curvature K^{ab} of the boundary.

The theory on the boundary can be seen to obtain its conformal invariance properties from two (related) sources: First, the metric on the boundary of the theory is not uniquely defined, since the AdS metric has a double pole there. It is instead only defined up to a conformal class of metrics. The double pole divergence of the metric is precisely what allows the theory to inherit the properties of a conformal field theory, since the pole shows up as the correct behaviour of the operator product expansion in the CFT [10]. Second, the $SO(n{+}1,2)$ isometry of the AdS$_{n+1}$ spacetime descends to the local conformal group of the field theory on the boundary [7].

The AdS/CFT correspondence is therefore a powerful way of studying properties of the dual conformal field theory, by relating them to properties of the gravity theory. Here, we will compute the action and stress tensor for certain gravity solutions and relate them to properties of their dual field theories. To deal with the divergences which appear in the gravitational action (arising from integrating over the infinite volume of spacetime), we shall use the "counterterm subtraction" method [14], which regulates the action by the addition of certain boundary counterterms which depend upon the geometrical properties of the boundary of the spacetime. They are chosen to diverge at the boundary in such a way as to cancel the bulk divergences [14,16] (see also refs. [17–21]):

$$\begin{aligned} I_{\rm ct} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^n x\sqrt{-h}\Bigg[\frac{(n-1)}{l} - \frac{l\mathcal{R}}{2(n-2)} + \\ \frac{l^3}{2(n-4)(n-2)^2}\left(\mathcal{R}_{ab}\mathcal{R}^{ab} - \frac{n}{4(n-1)}\mathcal{R}^2\right)\Bigg] \ . \end{aligned} \tag{4}$$

Here $\mathcal{R}$ and $\mathcal{R}_{ab}$ are the Ricci scalar and tensor for the boundary metric h. Using these counterterms one can construct a divergence–free stress tensor from the total action $I{=}I_{\rm bulk}{+}I_{\rm surf}{+}I_{\rm ct}$ by defining (see *e.g.* ref. [15]):

$$T^{ab} = \frac{2}{\sqrt{-h}}\frac{\delta I}{\delta h_{ab}} \ . \tag{5}$$

For orientation, a metric on AdS$_{n+1}$, in global coordinates is:

$$ds^2 = -\left(1+\frac{r^2}{l^2}\right)dt^2 + \frac{dr^2}{(1+r^2/l^2)} + r^2 d\Omega^2_{n-1} \ , \tag{6}$$

where $d\Omega^2_{n-1}$ is the metric on a round S^{n-1}. Equation (5) gives a definition of the action and stress–tensor on any region (radius r in the coordinates that we will choose later) bounding the interior of AdS$_{n+1}$. The AdS/CFT relation equates these quantities to a dual conformal field theory residing "on the boundary" at ($r{\to}\infty$).

As stated before, the metric restricted to the boundary, h_{ab}, diverges due to an infinite conformal factor, which is r^2/l^2. We take the background metric upon which the dual field theory resides as

$$\gamma_{ab} = \lim_{r\to\infty}\frac{l^2}{r^2}h_{ab} \ . \tag{7}$$

and so the field theory's stress–tensor, $\widehat{T}^{ab}$, is related to the one in (5) by the rescaling [22]:

$$\sqrt{-\gamma}\,\gamma_{ab}\widehat{T}^{bc} = \lim_{r\to\infty}\sqrt{-h}\,h_{ab}T^{bc} \ . \tag{8}$$

This amounts to multiplying all expressions for T^{ab} displayed later by $(r/l)^{n-2}$ before taking the limit $r{\to}\infty$. For the AdS$_{n+1}$ example (6), the n dimensional boundary upon which the theory resides is the Einstein universe, with metric $ds^2 = -dt^2 + l^2 d\Omega^2_{n-1}$. In the case of $n{=}4$, the field theory stress tensor computed using the above methods [14] can be written (as for other n) in the standard perfect fluid form [22] ($u_a{=}(1,0,0,0)$):

$$\widehat{T}_{ab} = \frac{1}{64\pi l G}\left(4u_a u_b + \gamma_{ab}\right) = \frac{N^2}{32\pi^2 l^4}\left(4u_a u_b + \gamma_{ab}\right). \tag{9}$$

Here we used the dictionary [7] between gravity and field theory quantities, $G{=}l^3\pi/2N^2$. The total energy, $E{=}\int d^3x\widehat{T}_{00}{=}3N^2/16l$, is in fact [14] the Casimir energy of the $\mathcal{N}{=}4$ supersymmetric $SU(N)$ Yang–Mills theory on the S^3. (See also ref. [23].) The conformal invariance of the theory is evident in the fact that $\widehat{T}^a_a{=}0$. It is worth stressing that there should be no confusion about the fact that the theory is conformal while in a box, S^3, which has a scale, l. Conformal invariance is preserved since this scale enters in a conformally invariant way in the metric itself.

This prescription gives a method for computing the stress tensor of a large class of field theories, which may be obtained by studying spacetimes which are asymptotically locally AdS. We will now show that it also provides

a natural method for generating examples of field theories which are scale but not conformally invariant, by placing a conformal field theory on a spacetime with just the correct properties we asked for earlier. We shall present two examples here, and will report more details and examples in an extended publication [24].

Our examples come from the Kerr–AdS spacetimes in five and seven dimensions (with only one of the two rotation parameters non–zero [25]):

$$
\begin{aligned}
ds^2 = &-\frac{\Delta_r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2\\
&+\frac{\Delta_\theta\sin^2\theta}{\rho^2}\left(adt - \frac{(r^2+a^2)}{\Xi}d\phi\right)^2\\
&+r^2\cos^2\theta d\Omega^2_{n-3}\ ,
\end{aligned}
$$

$$
\begin{aligned}
&\Xi = 1-a^2/l^2\ , \quad \rho^2 = r^2 + a^2\cos^2\theta\ ,\\
&\Delta_r = (r^2+a^2)(1+r^2/l^2) - 2MGr^{4-n}\ ,\\
&\Delta_\theta = 1-(a^2/l^2)\cos^2\theta
\end{aligned}
\tag{10}
$$

where n=4 or 6, and:

$$
d\Omega^2_1 = d\psi^2\ ;\ \ d\Omega^2_3 = d\psi^2 + \sin^2\psi d\eta^2 + \cos^2\psi d\beta^2\ . \tag{11}
$$

Using the prescription (7), the metric on which the field theory (either the $\mathcal{N}$=4 Yang–Mills theory for n=4 or the (0,2) CFT for n=6) resides can be seen to be that of a rotating Einstein universe [27] (see also refs. [26,28–30]):

$$
\begin{aligned}
ds^2 = -dt^2 + \frac{2a\sin^2\theta}{\Xi}dtd\phi + l^2\frac{d\theta^2}{\Delta_\theta}&\\
+l^2\frac{\sin^2\theta}{\Xi}d\phi^2 + l^2\cos^2\theta d\Omega^2_{n-3}&\ .
\end{aligned}
\tag{12}
$$

The gauge theory stress tensor of the Kerr–AdS$_5$ spacetime was computed in ref. [31], and the full expression can be found there. In particular the trace is:

$$
\widehat{T}^a_a = -\frac{N^2a^2}{4\pi^2l^6}[a^2/l^2(3\cos^4\theta - 2\cos^2\theta) - \cos 2\theta]\ . \tag{13}
$$

While it is non–zero, a quick computation shows that this is a total derivative, and it is in fact $\widehat{T}^a_a = -(N^2/\pi^2)E_4$, where the Euler density is:

$$
E_4 \equiv \frac{1}{64}\left[\mathcal{R}^{\mu\nu\kappa\lambda}\mathcal{R}_{\mu\nu\kappa\lambda} - 4\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} + \mathcal{R}^2\right]\ . \tag{14}
$$

The coefficient is precisely the field theory value [17]. Since $\int d^4x\sqrt{-\gamma}\,\widehat{T}^a_a$=0, (which fits that the action, I, is computed to be finite [31,25]), this is our first example of having preserved scale invariance while having broken conformal invariance. This was considered as a possibility in ref. [31]. However, it was noticed there that the trace was also proportional to $\Box\mathcal{R}$. So in fact, it was misidentified there with an anomaly of type D. A proposal was made there to add an $\mathcal{R}^2$ counterterm to the action and define an "improved" [1] stress tensor (see also refs. [32]). However, as pointed out in a note–in–proof added to ref. [31], it was not possible to do this while preserving the values for the physical conserved quantities (for example) like angular momentum. The point we stress here is that it is quite a *special* case that the Euler density happened to be proportional to $\Box\mathcal{R}$. In general it cannot be written in this form. Our anomaly is purely of type A, and as such its coefficient cannot be changed by adding counterterms. Instead, we accept the presence of the anomaly and give up conformal invariance; the rotation parameter a has broken conformal invariance of the theory, but scale invariance is preserved.

We now turn to the case of Kerr–AdS$_7$. We computed the non–vanishing components for the stress tensor at large r, to $O(r^{-4})$ using eqns. (4) and (5):

$$
\begin{aligned}
T_{tt} = &\frac{l^3}{640\pi Gr^4}\left[(1+a^2/l^2)(-131a^2/l^2 - 423a^4/l^4\cos^4\theta)\right.\\
&+219(1+a^4/l^4)a^2/l^2\cos^2\theta + 25(1+a^6/l^6)\\
&\left.+235a^6/l^6\cos^6\theta + 400MG/l^4 + 492a^4/l^4\cos^2\theta\right] + \cdots\ ,\\
T_{t\phi} = &-\frac{l^3a\sin^2\theta}{640\pi Gr^4\Xi}\left[-231a^6/l^6\cos^4\theta + 5 + a^2/l^2\right.\\
&+55a^6/l^6\cos^6\theta + 189a^6/l^6\cos^2\theta - 25a^6/l^6\\
&+168a^4\cos^2\theta/l^4 - 51a^4/l^4\cos^4\theta - 101a^4/l^4\\
&\left.-3a^2/l^2\cos^2\theta + 400MG/l^4\right] + \cdots,\\
T_{\phi\phi} = &\frac{al^5\sin^2\theta}{640\pi Gr^4\Xi^2}\left[102a^4/l^4 + 51a^4/l^4\cos^4\theta\right.\\
&-55a^6\Xi/l^6\cos^6\theta - 171a^4/l^4\cos^2\theta + 189a^8l^8\cos^2\theta/\\
&+3a^2/l^2\cos^2\theta + 4a^2/l^2 - 231a^8/l^8\cos^4\theta\\
&180a^6/l^6\cos^4\theta + 480MGa^2/l^6\cos^2\theta\\
&+80MG/l^4 - 21a^6\cos^2\theta/l^6 - 76a^6/l^6 - 25a^8/l^8\\
&\left.+400a^2MG/l^6 - 5\right] + \cdots\ ,\\
T_{\theta\theta} = &-\frac{l^5}{640\pi G\Delta_\theta r^4}\left[5a^2\Xi/l^2 - 80MG/l^4\right.\\
&+3a^2/l^2\cos^2\theta(1+a^4/l^4) + 66a^4/l^4\cos^2\theta\\
&\left.-45a^4/l^4\cos^4\theta(1+a^2/l^2) + 25a^6/l^6\cos^6\right] + \cdots\ ,\\
T_{\psi\psi} = &-\frac{l^5\cos^2\theta}{640\pi Gr^4}\left[5\Xi(1-a^4/l^4) - 80MG/l^4\right.\\
&-51a^4/l^4\cos^4\theta(1+a^2/l^2) - 3a^2/l^2\cos^2\theta(1+a^4/l^4)\\
&\left.+46a^4/l^4\cos^2\theta\right] + \cdots\ ,\\
T_{\eta\eta} = &\sin^2\psi T_{\psi\psi}, \quad T_{\beta\beta} = \cos^2\psi T_{\psi\psi}.
\end{aligned}
\tag{15}
$$

A computation shows that the action is again finite:

$$
\begin{aligned}
I = &\frac{2\pi^3(r_+^2+a^2)r_+}{G\Xi\left(3r_+^4/l^2 + 2r_+^2(1+a^2/l^2) + a^2\right)}\left[160(r_+^6/l^6\right.\\
&\left.+r_+^4/l^6a^2 - M/l^4) + 5a^4/l^4 + 50\Xi + a^6/l^6\right]\ ,
\end{aligned}
\tag{16}
$$

where r_+ is the location of the horizon, the largest root of Δ_r=0.

Taking the trace of the stress tensor yields:

$$\widehat{T}^a_a = -\frac{a^2N^3}{2\pi^3 l^8}\left[5a^4/l^4\cos^6\theta - 8\cos^4\theta a^2/l^2(1+a^2/l^2)\right.$$
$$\left.+3\cos^2\theta(1+a^4/l^4+3a^2/l^2) - 2(1+a^2/l^2)\right] \quad (17)$$

where we used the relation [7] $N^3{=}3\pi^2l^5/16G$ between the gauge theory and the gravity parameters.

Again, we see that this trace is a total derivative. Furthermore, $\widehat{T}^a_a = -(N^3/4508\pi^3)E_6$. (The Euler density E_6 is displayed in *e.g.* ref. [33], where the CFT is discussed at weak coupling). The coefficient matches the results in refs. [17,20,33]. (*n.b.,* a typo in ref. [17] is corrected in ref. [33].) Just as in the four dimensional case, we see that for this special situation, the Euler density can be written in terms of type D quantities. In the notation of ref. [33], it is of the form $\sum_{i=1}^7 d_iC_i$, with d_5 and d_7 zero, since they depend on the Weyl tensor, and $\{d_1,d_2,d_3,d_4,d_6\} = \{0, 1/9, 1/72, -5/12, 1/72\}$. (A useful parameterisation, but of course, not unique.) This completes our second counterexample to the folklore.

In conclusion, we have used the AdS/CFT correspondence to define known four and six dimensional conformal field theories, at large N, on spacetimes with properties chosen so that conformal invariance, but not scale invariance, is broken. The crucial point is that there are "many faces" [16] to anti–de Sitter spacetime, which can be found by slicing it in different ways by various coordinate choices, and then picking the boundary for the field theory to live on. The Kerr–AdS solutions give a particularly interesting slicing, for our present purposes: The $M{=}0$ limit is just AdS in very non–standard coordinates. The boundary is related to the standard static Einstein universe (which has vanishing Euler density) by a complicated change of variables, and a conformal transformation [25]. This is why the Euler density can be globally written as a total derivative, while the manifold remains conformally flat. It would be interesting to characterise such spacetimes further, since they give a straightforward means of defining the sort of field theory examples discussed herein.

Acknowledgements: We would like to thank Vijay Balasubramanian, Roberto Emparan, Juan Maldacena, Rob Myers, Joe Polchinski, and Al Shapere for comments and discussions, and especially Paul Townsend for raising the main issue discussed in this paper. CVJ would like to thank the Theory group at Harvard University for hospitality while this paper was written. AMA's work was supported by an NSF Career grant, #PHY–9733173. This paper is report #'s UK/00–02, and DTP/00/43. This paper was brought to you by the letters A, d, S, C, F, and T, and the numbers 4, 5, 6 and 7.

[1] C.G. Callan, S. Coleman and R. Jackiw, Ann. Phys. **59**, 42 (1970).
[2] S. Coleman and R. Jackiw, Annals Phys. **67**, 552 (1971).
[3] C. M. Hull and P. K. Townsend, Nucl. Phys. **B274**, 349 (1986).
[4] J. Polchinski, Nucl. Phys. **B303**, 226 (1988).
[5] A. Iorio, L. O'Raifeartaigh, I. Sachs, C. Wiesendanger, Nucl. Phys. **B495** (1997) 433, hep-th/9607110.
[6] S. Deser and A. Schwimmer, Phys. Lett. **B309** (1993) 279, hep-th/9302047.
[7] J. Maldacena, Adv. Theor. Math. Phys. **2** 231 (1998), hep-th/9711200.
[8] J. Polchinski, S. Chaudhuri and C. V. Johnson, hep-th/9602052; J. Polchinski, hep-th/9611050.
[9] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. **323**, 183 (2000), hep-th/9905111.
[10] E. Witten, Adv. Theor. Math. Phys. **2** 253 (1998), hep-th/9802150.
[11] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. **B428** 105 (1998), hep-th/9802109.
[12] G. W. Gibbons and S. W. Hawking, Phys. Rev **D15**, 2752 (1977).
[13] S. W. Hawking and G. T. Horowitz, Class. Quant. Grav. **13**, 1487 (1996) gr-qc/9501014.
[14] V. Balasubramanian and P. Kraus, Commun. Math. Phys. **208**, 413 (1999), hep-th/9902121.
[15] J. D. Brown and J. W. York, Phys. Rev. **D47**, 1407 (1993).
[16] R. Emparan, C. V. Johnson and R. C. Myers, Phys. Rev. **D60**, 104001 (1999), hep-th/9903238.
[17] M. Henningson and K. Skenderis, J.H.E.P. 9807 023 (1998), hep-th/9806087.
[18] S. R. Lau, Phys. Rev. **D60** (1999) 104034, gr-qc/9903038.
[19] R. B. Mann, Phys.Rev. **D60** (1999) 104047, hep-th/9903229.
[20] P. Kraus, F. Larsen and R. Siebelink, Nucl. Phys. **B563**, 259 (1999), hep-th/9906127.
[21] S. N. Solodukhin, hep-th/9909197.
[22] R.C. Myers, hep-th/9903203.
[23] G. T. Horowitz and R. C. Myers, Phys. Rev. **D59**, 026005 (1999) hep-th/9808079.
[24] Adel M. Awad and C. V. Johnson, to appear.
[25] S. W. Hawking, C. J. Hunter and M. M. Taylor–Robinson, Phys. Rev. **D59** (1999) 064005, hep-th/9811056.
[26] D. Klemm, JHEP **9811**, 019 (1998), hep-th/9811126.
[27] M.J. Cassidy and S.W. Hawking, Phys. Rev. **D57**, 2372 (1998) hep-th/9709066.
[28] D. S. Berman and M. K. Parikh, Phys. Lett. **B463**, 168 (1999), hep-th/9907003.
[29] S. W. Hawking and H. S. Reall, Phys. Rev. **D61**, 024014 (2000), hep-th/9908109.
[30] K. Landsteiner, E. Lopez, JHEP 9912 (1999) 020, hep-th/9911124.
[31] A. Awad and C. V. Johnson, Phys. Rev. **D61** (2000) 084025, hep-th/9910040.
[32] J. Ho, hep-th/9910124 and hep-th/0005250.
[33] F. Bastianelli, S. Frolov and A. A. Tseytlin, JHEP **0002**, 013 (2000), hep-th/0001041.